Diffusion in a two-dimensional anisotropic web map by extrinsic noise applied to the intrinsically perturbed quantity

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Diffusion by an extrinsic noise in a two-dimensional anisotropic web mapping is studied in the case where an extrinsic noise is applied to the intrinsically perturbed (intrinsically active) physical quantity and the intrinsic web diffusion is negligible. Contrary to the case where the extrinsic noise is applied to the other (intrinsically passive) physical quantity to yield a highly anisotropic diffusion scaling [Gunyoung Park and C. S. Chang, Phys. Rev. E **64**, 026211 (2001)], the diffusion scaling in this case is found to be isotropic.

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I. INTRODUCTION

Understanding the effect of extrinsic noise on global diffusion is an important problem in nonlinear dynamical systems. In the so-called standard mapping, this problem has been relatively simple and well understood [1–3]: Below the stochasticity threshold, the extrinsic noise generates global diffusion by letting the phase points ''leak'' across the Kol'mogorov-Arnol'd-Moser (KAM) curves in proportion to the extrinsic noise strength [2]. Above the stochasticity threshold, on the other hand, the extrinsic noise reduces the global diffusion by trapping the phase points in the KAM islands [3]. The amount of reduction by the extrinsic noise in the stochastic diffusion rate is the ratio of phase-space areas intrinsic to extrinsic stochasticity. This rate of reduction has been observed to be a universal phenomenon, regardless of the mapping type [3–5].

In the present work it will be assumed that the intrinsic perturbation is much below the stochasticity threshold, hence, the diffusion is entirely from extrinsic noise. In an isotropic web mapping, which arises when a linear oscillator is resonantly perturbed, it was shown in Ref. [4] that the diffusion is isotropic and proportional to the extrinsic noise strength l and square root of the intrinsic perturbation strength K (thus, $D \propto l\sqrt{K}$) if the extrinsic noise l is weaker than the intrinsic perturbation K, and $D \propto l^2$ if $l > K$.

In an anisotropic two-dimensional web mapping, on the other hand, it is reasonable to expect that global diffusion by extrinsic noise is anisotropic. Reference [5] indeed found out that the global diffusion in an anisotropic web map is highly anisotropic when the intrinsic and extrinsic noises do not act on the same physical quantity. In this case, diffusion in the extrinsically perturbed direction shows the same scaling as that in the isotropic web mapping, i.e., proportional to $l\sqrt{K}$ for $l < K$ and to l^2 for $l > K$. On the other hand, diffusion in the intrinsically perturbed direction is different: it is proportional to $lK^{3/2}$ if $l < K$ and proportional to K^2 (independent of l) if $l > K$.

In the present work, we study an anisotropic web map in the case where the extrinsic noise and intrinsic perturbation exist on the same physical quantity [6]. The present numerical simulation finds that the diffusion becomes isotropic in this case, regardless of the extrinsic noise strength.

The paper is organized as follows. In Sec. II, an anisotropic web map is defined with an extrinsic noise existing on the same physical variable as the intrinsic perturbation does. In Sec. III, a detailed numerical simulation of the diffusion rates is presented, which finds that the global diffusion is isotropic. A simple analytic explanation of the surprising result is also presented. Conclusion and discussion are presented in Sec. IV.

II. A NOISY ANISOTROPIC WEB MAP

When we add a toggle factor $(-1)^n$ to the so-called standard mapping, we obtain an area-preserving anisotropic web mapping [5]

$$P_{n+1} = P_n + K \sin \varphi_n ,$$

$$\varphi_{n+1} = \varphi_n + (-1)^n P_{n+1}$$

for two variables P_n and φ_n with the intrinsic perturbation parameter K. We call P the intrinsically active quantity and φ the passive quantity.

Owing to the ''toggle'' factor in the second equation, we call it toggle mapping. When the toggle factor is replaced by unity the standard mapping is restored. Figure 1 shows the phase-space structure of this mapping for even n numbers. Odd n numbers form a similar phase-space structure at a different φ location. The phase space is divided into infinitely periodic, two-dimensional, anisotropic tiles. The boundary between the tiles form a connected web structure (separatrix network). Within a tile, the phase points rotate along the closed KAM curves. Properties of the intrinsic glo-

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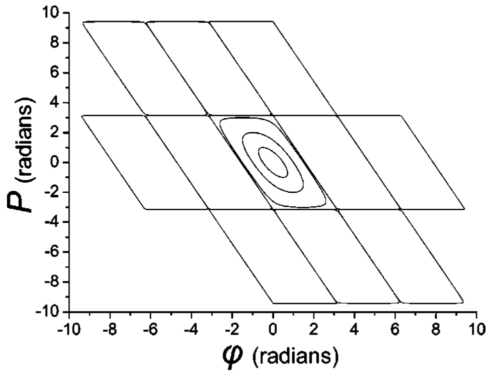

FIG. 1. Phase-space structure of Eq. (2) for $l=0$ in the limit $K\to 0$, where connected separatrix network is given by $\cos\varphi + \cos(\varphi+P)=0$. P and φ are in radians.

bal web diffusion in the toggle mapping for large enough K is not much different from those of the usual isotropic web mapping [5].

Reference [5] studied the extrinsic noise effect on this mapping when the extrinsic noise explicitly scatters the passive quantity φ:

$$P_{n+1}=P_n+K\sin\varphi_n,$$

$$\varphi_{n+1}=\varphi_n+(-1)^n P_{n+1}+l\xi, \tag{1}$$

where $l\xi$ is the extrinsic noise, l is the noise strength, and ξ is taken to be a normal distribution of random numbers with variance $=1$ and mean $=0$. As summarized in the preceding section, Ref. [5] found out that the global diffusion is highly anisotropic. This finding presented no surprise since the mapping itself is highly anisotropic.

The extrinsic noise can also enter in the active (P) direction in a real physical situation [6]. In this case the noisy toggle mapping can be represented by

$$P_{n+1}=P_n+K\sin\varphi_n+l\xi,$$

$$\varphi_{n+1}=\varphi_n+(-1)^n P_{n+1}. \tag{2}$$

In the present study we use the numerical technique of Ref. [5] to evaluate diffusion coefficients from Eq. (2).

III. DIFFUSION COEFFICIENTS

The discrete mapping, Eq. (2), is studied numerically. In order to measure the diffusion coefficient, a single orbit is broken into N segments, each of which has T mapping steps. Thus, the total length of the whole orbit is NT. Numerical diffusion coefficient for P (or φ) is then given by

$$D_P=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{2T}(P_i-P_{i-1})^2. \tag{3}$$

Here, T must be sufficiently large to ensure that the transient behavior dies out, and N must be large enough to provide meaningful statistics. As demonstrated in the Ref. [5] we find that $T=1000$ and $N=2\times10^6$ are proper. The dimension of D_P (or D_φ) is [radian^2/mapping step].

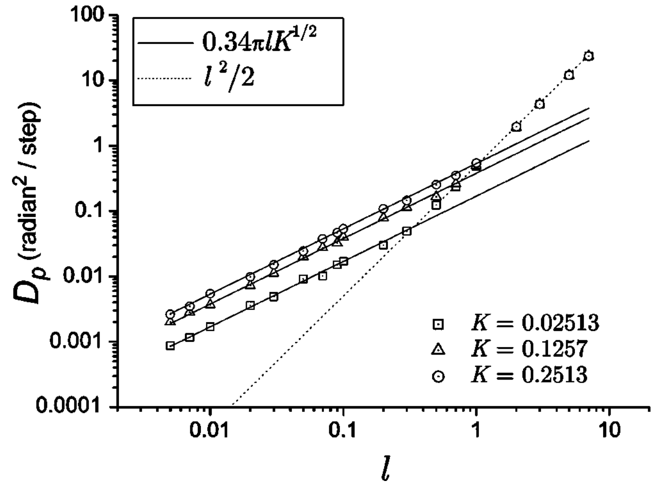

FIG. 2. D_p(radian^2/mapping step) vs the dimensionless noise strength l. Solid lines represent $D_p=0.34\pi l\sqrt{K}$ and dotted line represents $D_p=l^2/2$.

Figures 2 and 3 show numerically evaluated diffusion coefficient D_P in terms of the extrinsic noise strength l and the internal perturbation strength K, respectively. Figures 4 and 5 show numerical diffusion coefficient D_φ in terms of the extrinsic noise strength l and the internal perturbation strength K. D_P and D_φ show identical behavior (isotropic) except for the factor-of-2 difference ($D_P=2D_\varphi$). The factor-of-2 difference does not have a physical significance since the variables can be uniformly rescaled to remove it. Furthermore, the diffusion exhibits the same scaling law as in the isotropic web map [4], which is $D\propto l\sqrt{K}$ for small $l\ll K^{1/2}$ and $D\propto l^2$ for large $l\gg K^{1/2}$.

This result is remarkably different from the case where the extrinsic noise is in the passive (φ) direction. In that case the diffusion is highly anisotropic [5]: $D_P\sim lK^{3/2}$ and $D_\varphi\sim lK^{1/2}$ at low l; and $D_P\propto K^2$ and $D_\varphi\propto l^2$ at high l. By simply

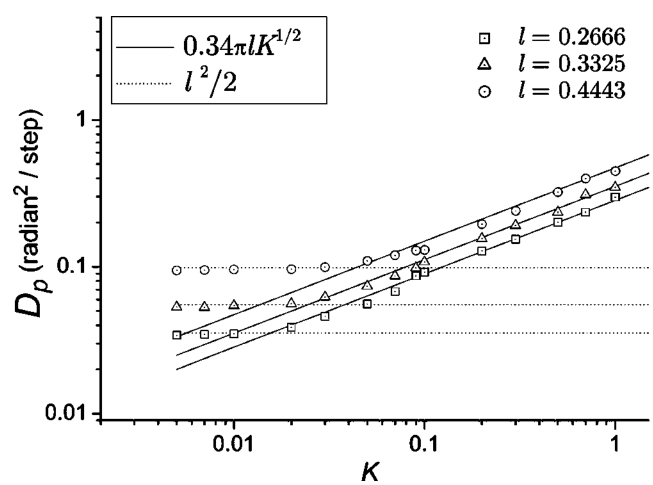

FIG. 3. D_p(radian^2/mapping step) vs the dimensionless intrinsic perturbation strength K. Solid lines represent $D_p=0.34\pi l\sqrt{K}$ and dotted lines represent $D_p=l^2/2$.

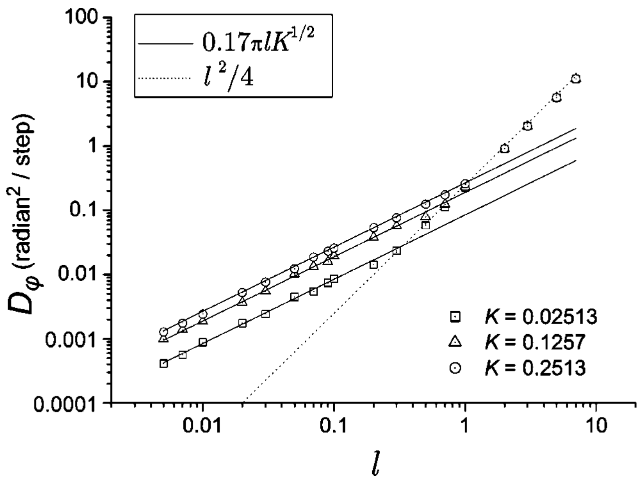

FIG. 4. D_φ(radian^2/mapping step) vs the dimensionless noise strength l. Solid lines represent $D_p = 0.17\pi l\sqrt{K}$ and the dotted line represents $D_\varphi = l^2/4$.

moving the extrinsic noise to the intrinsically active (P) direction, the extrinsic-noise driven diffusion of the anisotropic web map becomes basically the same as that of the isotropic map.

We find that this interesting behavior can be easily understood by a simple transformation of the mapping equation (2) into the two-step mapping equation

$$P_{n+2} = P_n + K\sin\varphi_n + K\sin(\varphi_n + P_n + K\sin\varphi_n + l\xi_1) + l\xi_1 + l\xi_2,$$

$$\varphi_{n+2} = \varphi_n - K\sin(\varphi_n + K\sin\varphi_n + P_n + l\xi_1) - l\xi_2. \quad (4)$$

The one-step mapping equation (2) produces two separate phase plots at two different φ locations, one corresponding to even n and the other to odd n. By taking a two-step difference in Eq. (4), we have isolated the phase plot corresponding to even n numbers, as shown in Fig. 1 for $l = 0$.

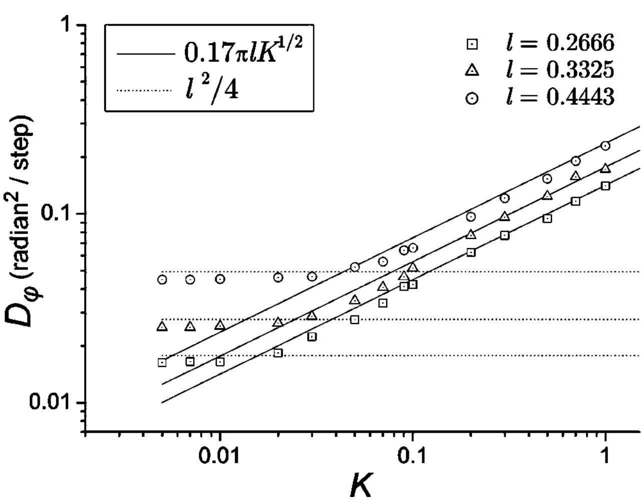

FIG. 5. D_φ(radian^2/mapping step) vs the dimensionless intrinsic perturbation strength K. Solid lines represent $D_p = 0.17\pi l\sqrt{K}$ and dotted lines represent $D_\varphi = l^2/4$.

TABLE I. Comparison between the extrinsic-noise-driven diffusion coefficients. The intrinsic perturbation is on P.

	Extrinsic noise on φ		Extrinsic noise on P	
	$l \gg K$	$l \ll K$	$l \gg K$	$l \ll K$
D_p	$K^2/4$	$0.08\pi l K^{3/2}$	$l^2/2$	$0.34\pi l\sqrt{K}$
D_φ	$l^2/2$	$0.20\pi l\sqrt{K}$	$l^2/4$	$0.17\pi l\sqrt{K}$

The K dependent terms produce KAM rotations and separatrix structure of Fig. 1 and the l dependent terms induce the tile-to-tile diffusion.

Notice here that the extrinsic noise, which was originally introduced in the active P direction in Eq. (2), propagated into the passive φ-direction in Eq. (4). Under the present assumption of small K, the l terms within the K-proportional terms play negligible contribution compared to the standalone l terms. Since ξ_1 and ξ_2 are two independent random numbers, $l(\xi_1 + \xi_2)$ in $P_{n+2} - P_n$ and $l\xi_2$ in $\varphi_{n+2} - \varphi_n$ play the same physical roles. Only the magnitudes are different (by $\sqrt{2}$).

This behavior of the extrinsic noise is in contrast with the case studied in Ref. [5] where the extrinsic noise originally entered in the passive φ direction. The two-step mapping equation in that case becomes, from Eq. (1),

$$P_{n+2} = P_n + K\sin\varphi_n + K\sin(\varphi_n + P_n + K\sin\varphi_n + l\xi_1),$$

$$\varphi_{n+2} = \varphi_n - K\sin(\varphi_n + P_n + K\sin\varphi_n + l\xi_1) + l(\xi_1 + \xi_2). \quad (5)$$

The lowest-order extrinsic noise is confined to φ only, in sharp contrast to Eq. (4). Hence, Eq. (4) is basically isotropic with respect to the extrinsic noise, while Eq. (5) is inherently anisotropic.

In the direction of extrinsic noise (φ), Ref. [5] showed that the diffusion obeys noisy scaling: $D \propto l\sqrt{K}$ for small $l \ll K^{1/2}$ and $D \propto l^2$ for large $l \gg K^{1/2}$. In the present case, the extrinsic noise is active in both directions. Thus, the diffusion follows the noisy scaling in both directions, yielding an isotropic diffusion scaling.

IV. CONCLUSION AND DISCUSSION

We have shown that an extrinsically driven diffusion from a two-dimensional anisotropic web mapping can be different depending upon how the extrinsic noise enters into the system. If the extrinsic noise is applied to the passive quantity, the noise does not propagate into the active quantity. The diffusion is then highly anisotropic. However, if the extrinsic noise is applied to the intrinsically active quantity, the noise propagates into the passive quantity. Diffusion in this case becomes isotropic, following a noisy scaling of Ref. [4].

In other words, an extrinsic noise in the passive direction makes only a small contribution to the diffusion in the intrinsically active direction. However, an extrinsic noise in the active direction drives a large diffusion in the passive direction. The diffusion coefficients are summarized and compared in Table I.

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